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Quantum corrections for (anti)-evaporating black hole

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ABSTRACT

In this paper we analyse the quantum correction for Schwarzchild black hole in the Unruh state in the framework of spherically symmetric gravity (SSG) model. SSG is two-dimensional dilaton model which is obtained by spherically symmetric reduction from four-dimensional theory. We find the one-loop geometry of the (anti)-evaporating black hole and corrections for mass, entropy and apparent horizon.

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1 Introduction

Two-dimensional spherically symmetric gravity model (SSG) is interesting for many reasons. This model is obtained from four dimensional (4D) Einstein-Hilbert action coupled minimally to scalar fields by spherically symmetric reduction of metric and scalar fields. The reduction is done in the spirit of string theory, via the introduction of dilaton field Φ, assuming that the line element is of the form:

$$ds^2_{(4)} = g_{\mu\nu} dx^\mu dx^\nu + e^{-2\Phi}(\sin^2\theta d\phi^2 + d\theta^2) \ , \tag{1}$$

where $\mu, \nu = 0, 1$. The action for this model is given by the equation ($^{eq:S2]}$) below, and it has Schwarzchild black hole as a static vacuum solution.

One reason which makes this model interesting is that the quantum effective action for scalar fields can be calculated to the one-loop order. This gives the possibility to obtain the backreaction effects of quantized matter to gravity analytically (in the case of black hole solution this is the backreaction of the Hawking radiation). These analytic 2D calculations can then be compared with the numerical 4D estimates, as the effective action can not be obtained analytically in 4D. This analysis was done in many details for the Hartle-Hawking vacuum state of matter [1, 2, 3]. Summarizing, one can say that the main drawback of the SSG model is that it gives the negative luminosity of the black hole. It is argued in the literature [4] that this result is a consequence of the fact that only the radial modes of the scalar field are counted in the expectation value of the energy density while the angular modes are omitted. Formally, the negative luminosity is not a surprising result as the scalar field and the dilaton are strongly coupled at spatial infinity, as can be seen from the action ($^{eq:S2]}$). There are also some attempts to improve the lagrangian of the model [5, 6, 7, 8].

2D dilaton gravity is also interesting by itself from the heuristic point of view. Dilaton couplings are present in all theories which are obtained by dimensional reduction from string theories. Furthermore, the one-loop effective actions are nonlocal. One possibility to deal with such actions is their conversion to the local form by introduction of auxiliary fields. The local form of action is rather handy for calculations (e.g., for equations of motion or energy-momentum tensor). On the other hand, the fact that auxiliary fields describe nonlocal effects implies that they are dynamical, and it is a priori unclear how to fix the arbitrary constants (or functions) in the solutions. It is also not known whether all solutions have the physical meaning. In the case of SSG model the properties of auxiliary fields are rather well established for the Hartle-Hawking vacuum state. In the present paper we extend the analysis to the Unruh vacuum. We think that it is of importance to understand the ways to describe nonlocal effects by auxiliary fields. SSG is important as it provides us with an example of the effective action which is tractable, but, as we shall see, in some respects more complicated than the (usually discussed) Polyakov-Liouville action.

The complementary way of discussing different vacuum states was developed in the very instructive paper [2] by Balbinot and Fabbri. Their analysis is based on the conformal properties of fields under the change of the conformal vacuum state. In

this method, the initial step is to identify the energy-momentum tensor (EMT) of one vacuum state (e.g., Boulware). Then one can find the expectation values of EMT in other states from conformal transformation properties of fields.

The organization of the paper is the following. In section 2 we solve the equations of motion for the auxiliary fields in the Unruh vacuum and obtain the value of energy-momentum tensor. In order to fix the arbitrary functions in the solution we use the conditions of regularity of EMT on the future horizon. For comparison, the energy momentum tensor is found by Balbinot-Fabbri procedure. The differences between Polyakov-Liouville action and SSG action are also discussed. In section 3 we find the influence of the Hawking radiation to the geometry in the one-loop order. In order to fix the integration constants in the metric, we impose the condition that the emitted flux of radiation is constant. We calculate the ADM mass of the black hole. In section 4 we obtain the position of the apparent horizon and entropy. Furthermore, we analyse the behaviour of the entropy along the line of the apparent horizon and find that the second law of thermodynamics is fulfilled.

2 Energy-momentum tensor and auxiliary fields

The Einstein-Hilbert action with minimally coupled N scalar fields, f_i $(i = 1, \dots N)$ in 4D is given by

$$\Gamma_0^{(4)} = \frac{1}{16\pi G}\int d^4x\sqrt{-g^{(4)}}R^{(4)} - \frac{1}{8\pi}\sum_i\int d^4x\sqrt{-g^{(4)}}(\nabla f_i)^2 \quad . \tag{2}$$

After spherically symmetric reduction ($^{reduc]}$), from the action ($^{eq:S4]}$) we get two-dimensional classical action Γ_0

$$\Gamma_0 \;=\; \frac{1}{4G}\int d^2x\sqrt{-g}\Big(e^{-2\Phi}(R + 2(\nabla\Phi)^2 + 2e^{2\Phi}) - 2Ge^{-2\Phi}\sum_i(\nabla f_i)^2\Big) \ , \tag{3}$$

where g and R denote two-dimensional metric and curvature. The Schwarzschild black hole is the classical vacuum solution of the equations of motion which follow from the action ($^{eq:S2]}$). This solution is given by

$$\begin{aligned} ds^2 &= -f(x^1)(dx^0)^2 + \frac{1}{f(x^1)}(dx^1)^2 \\ \Phi &= -\log x^1 \\ f_i &= 0 \ \text{ (except at the point } x^1 = 0) \ , \end{aligned} \tag{4}$$

where $f(x^1) = 1 - a/x^1$. The constant a is the radius of the event horizon, $a = 2MG$, and M is the mass of the Schwarzschild black hole.

When we add the one-loop quantum correction for the matter fields f_i to the classical action ($^{eq:S2]}$), we get the nonlocal effective action. Its one-loop part is given by [9, 10, 11, 12, 13, 14]:

$$\bar{\Gamma}_1 = -\frac{N}{96\pi}\int d^2x\sqrt{-g}\Big(R\frac{1}{\Box}R - 12R\frac{1}{\Box}(\nabla\Phi)^2 + 12R\Phi\Big) \ , \tag{5}$$

which describes the quantum effects of the scalar matter fields. Calculations can be simplified if the nonlocal correction part $\bar{\Gamma}_1$ is rewritten in the local form using two auxilliary fields ψ and χ [1]:

$$\Gamma_1 = -\frac{N}{96\pi}\int d^2x\sqrt{-g}\left[2R(\psi-6\chi)+(\nabla\psi)^2-12(\nabla\psi)(\nabla\chi)-12\psi(\nabla\Phi)^2+12R\Phi\right]. \tag{6}$$

The additional fields ψ and χ satisfy the equations of motion

$$\Box\psi = R\ , \tag{7}$$

$$\Box\chi = (\nabla\Phi)^2\ . \tag{8}$$

Γ_1 and $\bar{\Gamma}_1$ are equivalent in the following sense. If we introduce the equations (11]-22]) into the local form of the action Γ_1, we will get the nonlocal action $\bar{\Gamma}_1$ up to boundary terms [3]. This difference does not influence the equations of motion. The analysis of the boundary terms can be postponed till the calculation of ADM mass and it was done carefully [15].

The form of the action we will use is:

$$\begin{aligned}\Gamma = \Gamma_0+\Gamma_1 &= \frac{1}{4G}\int d^2x\sqrt{-g}\Big(r^2R+2(\nabla r)^2+2\Big)\\ &- \frac{\kappa}{4G}\int d^2x\sqrt{-g}\Big[(\nabla\psi)^2+2R\psi-12(\nabla\psi)(\nabla\chi)\\ &- 12\psi\frac{(\nabla r)^2}{r^2}-12R\chi-12R\log r\Big)\Big],\end{aligned} \tag{9}$$

where $\kappa = NG\hbar/24\pi$. Instead of the dilaton Φ we introduced new variable, $r=e^{-\Phi}$. Varying the action (eq:S]) we obtain the equations of motion [1]:

$$\Box\psi = R\ , \tag{10}$$

$$\Box\chi = \frac{(\nabla r)^2}{r^2}\ , \tag{11}$$

$$2\Box r - rR = -6\kappa\Big(2\psi\frac{\Box r}{r^2}+2\frac{(\nabla\psi)(\nabla r)}{r^2}-2\psi\frac{(\nabla r)^2}{r^3}+\frac{R}{r}\Big)\ , \tag{12}$$

$$\begin{aligned}&g_{\mu\nu}(\Box r^2-(\nabla r)^2-1)-2r\nabla_\mu\nabla_\nu r = 2GT_{\mu\nu} =\\ =\ &\kappa\Big(g_{\mu\nu}(2R+6\psi\frac{(\nabla r)^2}{r^2}-\frac{1}{2}(\nabla\psi)^2+6(\nabla\psi)(\nabla\chi)-12\frac{\Box r}{r})\\ +\ &\nabla_\mu\psi\nabla_\nu\psi-12\nabla_\mu\psi\nabla_\nu\chi-2\nabla_\mu\nabla_\nu\psi+12\nabla_\mu\nabla_\nu\chi\\ +\ &12\frac{\nabla_\mu\nabla_\nu r}{r}-12(1+\psi)\frac{\nabla_\mu r\nabla_\nu r}{r^2}\Big)\ .\end{aligned} \tag{13}$$

First, let us note that $r=x^1$ ($\Phi=-\log x^1$) remains to be the solution of the quantum-corrected equations of motion (eqpsi]-eqg]), so we see that the field r has the

[3]We would like to thank D. Vassilevich for the discussion considering this point.

meaning of radius. We will use the following notation for the coordinates: $x^1 = r$, $x^0 = t$.

We want to find the quantum correction of the geometry of 2D black hole for the case when the black hole evaporates. This means that black hole is in the Unruh state. Our calculation is perturbative in the orders of κ which is a small parameter. All quantities will be calculated to the first order in κ, as the effective action is also calculated to this precision only. The ansatz for the one-loop metric is

$$ds^2 = -F(r,\tilde{v})e^{2\kappa\varphi}d\tilde{v}^2 + 2e^{\kappa\varphi}d\tilde{v}dr \ , \tag{14}$$

and we solve the equations in Eddington-Finkelstein r, $\tilde{v}$ coordinates,

$$\tilde{v} = t + r_* = t + r + a\log\left(\frac{r}{a} - 1\right) \ . \tag{15}$$

The function F is taken in the form

$$F(r,\tilde{v}) = f(r) + \frac{\kappa m(r,\tilde{v})}{r} = 1 - \frac{a}{r} + \frac{\kappa m(r,\tilde{v})}{r} \ . \tag{16}$$

Introducing the ansatz ($^{eq:qs]}$) into ($^{eqr]\_eqg]}$), we get that the equations for unknown functions m and φ in the first order in κ take the simple form:

$$\kappa\partial_r\varphi = G\frac{T_{rr}}{r} \tag{17}$$

$$\kappa\partial_r m = 2Ge^{-\kappa\varphi}T_{r\tilde{v}} \tag{18}$$

$$\kappa\partial_{\tilde{v}} m = -2G(FT_{r\tilde{v}} + e^{-\kappa\varphi}T_{\tilde{v}\tilde{v}}) \ , \tag{19}$$

where $T_{rr}, T_{r\tilde{v}},$ and $T_{\tilde{v}\tilde{v}}$ are the corresponding components of the energy-momentum tensor defined by the equation ($^{eqg]}$). The EMT is a quantity of the first order in κ, so in order to determine it with the necessary precision we need the zero-th order solution for metric and auxiliary fields.

Let us briefly review how the solutions were found previously, in [1]. In the Hartle-Hawking state ψ and χ are time-independent, as they describe the black hole in thermal equilibrium with the Hawking radiation. Therefore, the solutions of the equations ($^{eqpsi]\_eqchi]}$) are

$$\psi = Cr + Ca\log\frac{r-a}{a} - \log\frac{r-a}{r} \quad , \tag{20}$$

$$\chi' = \frac{2Dr^2 - 2r + a}{2r(r-a)} \quad . \tag{21}$$

The assumption of regularity of EMT on the classical horizon $r = a$ in the free-falling frame gives the values of the integration constants: $C = \frac{1}{a}$, $D = \frac{1}{2a}$.

We will now solve the equations ($^{eqpsi]\_eqchi]}$) in the general case. As mentioned, we need the zero-th order metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -fd\tilde{v}^2 + 2d\tilde{v}dr \ . \tag{22}$$

The other quantities entering equations ($^{eqpsi]}$_$^{eqchi]}$) are

$$R = -\frac{d^2 f}{dr^2} \ , \quad \frac{(\nabla r)^2}{r^2} = \frac{f}{r^2} \ . \tag{23}$$

Introducing these values, the equation for ψ becomes

$$\Box\psi = \partial_r(2\partial_{\tilde{v}}\psi + f\partial_r\psi) = -\frac{d^2 f}{dr^2} \ , \tag{24}$$

and it reduces to the linear partial differential equation:

$$2\partial_{\tilde{v}}\psi + f\partial_r\psi = -\frac{df}{dr} + \tilde{\mathcal{G}}(\tilde{v}) \ . \tag{25}$$

In order to find the general solution of the equation ($^{psi1]}$) one has to find two independent integrals $\alpha(\tilde{v}, r, \psi) = \text{const}$ and $\beta(\tilde{v}, r, \psi) = \text{const}$ of the system

$$\frac{d\tilde{v}}{2} = \frac{dr}{f} = \frac{d\psi}{\tilde{\mathcal{G}}(\tilde{v}) - \partial_r f} \quad ; \tag{26}$$

the general solution of ($^{psi1]}$) is then an arbitrary function of α and β. In our case, the independent integrals are

$$\alpha = r_* - \frac{\tilde{v}}{2} \ , \quad \beta = \psi + \log f - \frac{1}{2}\int \tilde{\mathcal{G}}(\tilde{v})\, d\tilde{v} \quad . \tag{27}$$

Therefore, the general solution for ψ can be written in the form

$$\psi = -\log\left(1 - \frac{a}{r}\right) + \mathcal{G}(\tilde{v}) + \mathcal{C}(r_* - \frac{\tilde{v}}{2}) \ , \tag{28}$$

where $r_* = r + a\log\left(\frac{r}{a} - 1\right)$, while $\mathcal{G}(\tilde{v}) = \frac{1}{2}\int \tilde{\mathcal{G}}(\tilde{v})d\tilde{v}$ and $\mathcal{C}(r_* - \frac{\tilde{v}}{2})$ are arbitrary functions. Similarly, the equation for χ

$$\Box\chi = \partial_r(2\partial_{\tilde{v}}\chi + f\partial_r\chi) = \frac{f}{r^2} \ , \tag{29}$$

reduces to the system

$$\frac{d\tilde{v}}{2} = \frac{dr}{f} = \frac{d\chi}{\tilde{\mathcal{H}}(\tilde{v}) + \frac{a-2r}{2r^2}} \ . \tag{30}$$

The general solution for χ is

$$\chi = -\frac{1}{2}\log\frac{r(r-a)}{a^2} + \mathcal{H}(\tilde{v}) + \mathcal{D}(r_* - \frac{\tilde{v}}{2}) \ , \tag{31}$$

where $\mathcal{H}(\tilde{v})$ and $\mathcal{D}(r_* - \frac{\tilde{v}}{2})$ are arbitrary functions. The functions $\mathcal{G}(\tilde{v})$, $\mathcal{C}(r_* - \frac{\tilde{v}}{2})$, $\mathcal{H}(\tilde{v})$ and $\mathcal{D}(r_* - \frac{\tilde{v}}{2})$ describe various quantum states of matter. To recover the static Hartle-Hawking vacuum solution we have to put all functions linear in their arguments in order to cancel t-terms. This, combined with the condition of regularity gives $\mathcal{C}(r_* - \frac{\tilde{v}}{2}) = \frac{1}{a}(r_* - \frac{\tilde{v}}{2})$, $\mathcal{G}(\tilde{v}) = \frac{1}{2a}\tilde{v}$, $\mathcal{H}(\tilde{v}) = \frac{1}{4a}\tilde{v}$ and $\mathcal{D}(r_* - \frac{\tilde{v}}{2}) = \frac{1}{2a}(r_* - \frac{\tilde{v}}{2})$.

We now pass to the case of the Unruh vacuum. It is most naturally discussed in the null-coordinates u, v:

$$v = \tilde{v}\ , u = \tilde{v} - 2r_* = \tilde{v} - 2\left(r + a\log(\frac{r}{a} - 1)\right)\ . \tag{32}$$

The Unruh vacuum state is defined as the state which has the EMT regular on the future event horizon, $u \to \infty$, $v =$ constant. The conditions of regularity in the free falling frame read [16]:

$$T_{vv} < \infty\ ,\quad \frac{T_{uv}}{f} < \infty\ ,\quad \frac{T_{uu}}{f^2} < \infty\ . \tag{33}$$

Components of the energy-momentum tensor in the u, v coordinates can be found from the relations

$$T_{rr} = 4\left(\frac{r}{r-a}\right)^2 T_{uu} \tag{34}$$

$$T_{r\tilde{v}} = -2\frac{r}{r-a}(T_{uu} + T_{uv}) \tag{35}$$

$$T_{\tilde{v}\tilde{v}} = T_{uu} + 2T_{uv} + T_{vv} \tag{36}$$

Along with the condition of regularity of EMT, we will impose that at the spatial infinity $r \to \infty$ the outgoing flux T_{uu} has a constant nonvanishing value, while the ingoing flux T_{vv} tends to 0. When we introduce the solutions ($^{eq:gspsi]}$), ($^{eq:gschi]}$) for the components of EMT we get:

$$T_{uv} = \frac{a}{24\pi}\frac{r-a}{r^4} \tag{37}$$

$$\begin{aligned} T_{vv} &= \frac{(a-r)^2}{16\pi r^4}\log\frac{r-a}{r} + \frac{1}{48\pi}(\mathcal{G}'^2 - 12\mathcal{G}'\mathcal{H}' - 2\mathcal{G}'' + 12\mathcal{H}'') \\ &- \frac{1}{192\pi r^4}\Big(-3a^2 + 4ar + 12(a-r)^2\mathcal{C} + 12(a-r)^2\mathcal{G} \\ &+ (12ar^2 - 24r^3)\mathcal{G}'\Big) \end{aligned} \tag{38}$$

$$\begin{aligned} T_{uu} &= \frac{(a-r)^2}{16\pi r^4}\log\frac{r-a}{r} + \frac{1}{48\pi}(\mathcal{C}'^2 - 12\mathcal{C}'\mathcal{D}' - 2\mathcal{C}'' + 12\mathcal{D}'') \\ &- \frac{1}{192\pi r^4}\Big(-3a^2 + 4ar + 12(a-r)^2\mathcal{C} + 12(a-r)^2\mathcal{G} \\ &+ (6ar^2 - 12r^3)\mathcal{C}'\Big) \end{aligned} \tag{39}$$

(primes denote derivatives of the functions with respect to their arguments).

There is no information about the unknown functions contained in T_{uv}. Further, it can be seen that T_{vv} is regular on the horizon. The condition that $T_{vv} \to 0$ as $r \to \infty$ means that in this limit

$$\mathcal{G}'^2 - 12\mathcal{G}'\mathcal{H}' - 2\mathcal{G}'' + 12\mathcal{H}'' = 0\ . \tag{40}$$

The solution of the last equation, which is in accordance with the radiation law, is given by linear functions $\mathcal{G}$, $\mathcal{H}$:

$$\mathcal{G}(\tilde{v}) = g\,\tilde{v}\ , \quad \mathcal{H}(\tilde{v}) = h\,\tilde{v}\ , \tag{41}$$

with

$$g(g-12h) = 0\ , \tag{42}$$

i.e. either $g = 0$ or $g = 12h$.

Similarly, the condition that $T_{uu} \to \text{const}$ as $r \to \infty$ gives that the functions $\mathcal{C}$ and $\mathcal{D}$ are linear in their arguments,

$$\mathcal{C}(x) = c\,x\ , \quad \mathcal{D}(x) = d\,x\ . \tag{43}$$

Nonsingularity of $\frac{T_{uu}}{f^2}$ on the horizon gives us the values of the constants: $c = \frac{1}{a}$, $d = \frac{1}{2a}$. Introducing c and d in ($^{TUU]}$) we see that the luminosity has the Hartle-Hawking value $-\frac{5}{192\pi a^2}$. 2D black hole antievaporates. This is because we took into account the contribution of the $s-$modes of the radiation only.

To conclude our reasoning, let us observe that one arbitrariness remained, and that is the dependence of EMT on the constant g. This arbitrariness can be naturally fixed by choosing the $g = 0$ solution of the condition ($^{gh]}$). Note also that the value of the constant h does not enter EMT, and therefore we can fix it freely, e.g. $h = \frac{1}{4}$. Finally we have the solution for ψ, χ in the zero-th order

$$\psi = \frac{r}{a} + \log\frac{r}{a} - \frac{v}{2a}\ , \tag{44}$$

$$\chi = \frac{r}{2a} - \frac{1}{2}\log\frac{r}{a}\ . \tag{45}$$

We just mention briefly that it can be shown that for $g = 0$ the value of h does not influence the ADM mass.

We can now perform the Balbinot-Fabbri procedure [2] and compare the values of EMT. If the vacuum state of matter is defined in such a way that the ingoing and outgoing modes have positive frequency with respect to the coordinates u, v, the EMT corresponds to the Boulware state:

$$\left\langle u,v \right| \hat{T}_{uv} \left| u,v \right\rangle = -\frac{1}{12\pi}(\partial_v\partial_u\rho + 3\partial_v\Phi\partial_u\Phi - 3\partial_v\partial_u\Phi)\ , \tag{46}$$

$$\begin{aligned} \left\langle u,v \right| \hat{T}_{vv} \left| u,v \right\rangle &= -\frac{1}{12\pi}(\partial_v\rho\partial_v\rho - \partial_v^2\rho) + \frac{1}{2\pi}\Big(\rho(\partial_v\Phi)^2 + \frac{1}{2}\frac{\partial_v}{\partial_u}(\partial_v\Phi\partial_u\Phi)\Big) \\ &- \frac{1}{4\pi}(-2(\partial_v\rho)(\partial_v\Phi) + \partial_v^2\Phi)\ , \end{aligned} \tag{47}$$

$$
\begin{aligned}
\left\langle u,v\right| \hat{T}_{uu} \left|u,v\right\rangle &= -\frac{1}{12\pi}(\partial_u\rho\partial_u\rho - \partial_u^2\rho) + \frac{1}{2\pi}\Big(\rho(\partial_u\Phi)^2 + \frac{1}{2}\frac{\partial_u}{\partial_v}(\partial_+\Phi\partial_-\Phi)\Big) \\
&- \frac{1}{4\pi}(-2(\partial_u\rho)(\partial_u\Phi) + \partial_u^2\Phi) \ ,
\end{aligned} \tag{48}
$$

where $\rho = \frac{1}{2}\log(1-\frac{a}{r})$ is a conformal factor.

The conformal transformation to the other conformal state $|\tilde{u},\tilde{v}\rangle$ defined by the other set of null-coordinates $\tilde{u} = \tilde{u}(u)$, $\tilde{v} = \tilde{v}(v)$, gives

$$
\left\langle \tilde{u},\tilde{v}\right| \hat{T}_{uv} \left|\tilde{u},\tilde{v}\right\rangle = \left\langle u,v\right| \hat{T}_{uv} \left|u,v\right\rangle \ , \tag{49}
$$

$$
\begin{aligned}
\left\langle \tilde{u},\tilde{v}\right| \hat{T}_{vv} \left|\tilde{u},\tilde{v}\right\rangle &= \left\langle u,v\right| \hat{T}_{vv} \left|u,v\right\rangle + \frac{1}{24\pi}\left(\frac{G''}{G} - \frac{1}{2}\frac{G'^2}{G^2}\right) \\
&+ \frac{1}{4\pi}\left((\partial_v\Phi)^2\log(FG) + \frac{G'}{G}\int du\partial_v\Phi\partial_u\Phi\right) \ ,
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\left\langle \tilde{u},\tilde{v}\right| \hat{T}_{uu} \left|\tilde{u},\tilde{v}\right\rangle &= \left\langle u,v\right| \hat{T}_{uu} \left|u,v\right\rangle + \frac{1}{24\pi}(\frac{F''}{F} - \frac{1}{2}\frac{F'^2}{F^2}) \\
&+ \frac{1}{4\pi}\Big((\partial_u\Phi)^2\log(FG) + \frac{F'}{F}\int dv(\partial_u\Phi)(\partial_u\Phi)\Big) \ ,
\end{aligned} \tag{51}
$$

where $F(u) = \frac{du}{d\tilde{u}}$, $G(v) = \frac{dv}{d\tilde{v}}$.

Unruh vacuum state is the state $|U,v\rangle$, U being the Kruskal coordinate $U = -2ae^{\frac{u}{2a}}$. Using ($^{eq:EMTbf]\_eq:EMTbf1]}$) after simple calculation, we get the value of the EMT in the Unruh state ($\frac{1}{24\pi} = \frac{\kappa}{G}$) :

$$
T_{uv} = \frac{\kappa}{G}(1-\frac{a}{r})\frac{a}{r^3} \tag{52}
$$

$$
\begin{aligned}
T_{uu} &= \frac{\kappa}{G}\Big(\frac{3a^2-4ar}{8r^4} - \frac{5}{8a^2} - \frac{3}{2a}(\frac{a}{2r^2} - \frac{1}{r}) \\
&+ \frac{3}{2r^2}(1-\frac{a}{r})^2(\frac{v}{2a} - \frac{r}{a} - \log\frac{r}{a})\Big)
\end{aligned} \tag{53}
$$

$$
T_{vv} = \frac{\kappa}{G}\Big(\frac{3a^2-4ar}{8r^4} + \frac{3}{2r^2}(1-\frac{a}{r})^2(\frac{v}{2a} - \frac{r}{a} - \log\frac{r}{a})\Big) \ . \tag{54}
$$

These expressions are the same as the previously given ($^{TVV]\_TUU]}$) with fixed integration functions.

Let us give one final comment of the values of EMT ($^{tuv]\_tuu]}$). The obtained values have v-dependenence, i.e. t-dependence. This dependence does not show up

in the asymptotic behaviour of EMT and it was considered by [17] as an unwished property of the energy-momentum tensor. In fact, in [17] the auxiliary fields were constrained in such a way that the time-dependence of ψ, χ would not produce any time dependence in EMT. We think that a condition like this is too stringent and unneccessary. It holds, though, in the "minimal coupling" case, i.e. in the case when the effective action is given by the Polyakov-Liouville term only, as it can easily be seen. Namely, it is known [3] that in this case the change of the conformal frame produces in EMT only the additional term proportional to the Schwarzian derivative of the transformation of coordinates:

$$\left\langle \tilde{u},\tilde{v}\right| \hat{T}_{vv} \left|\tilde{u},\tilde{v}\right\rangle = \left\langle u,v\right| \hat{T}_{vv} \left|u,v\right\rangle + \frac{1}{24\pi}\Big(\frac{F''}{F} - \frac{1}{2}\frac{F'^2}{F^2}\Big) \ . \tag{55}$$

For exponential mappings, which are typical for the transformation to Kruskal coordinates, the Schwarzian derivative is constant. This means that if we start with the time-independent EMT for, e.g., Hartle-Hawking vacuum, we will get the time-independent EMT for all other conformal vacua. But this is the special property of the Polyakov-Liouville effective action. In SSG case the structure of the additional terms is more complicated and this brings the time-dependence in the Unruh vacuum state. The fact that this dependence is linear is in accordance with the expected property that the black hole in the Unruh vacuum radiates at constant rate, $\frac{dM(t)}{dt} =$ const. The meaning of the mass $M(t)$ will be discussed in more details after we solve the backreaction equations for the metric and identify the ADM mass of the solution.

3 Backreaction and corrected geometry

The equations which determine the one-loop correction of the metric can now easily be integrated. The solution is:

$$\varphi = \frac{5}{ar} + 3\frac{a-2v}{4ar^2} + \frac{3}{r^2}\log\frac{r}{a} - \frac{5}{2a^2}\log\frac{r}{l} + C_1 \tag{56}$$

$$\begin{aligned} m &= \frac{5r}{2a^2} + \frac{a+6v}{2ar} + \frac{11a-6v}{4r^2} - \frac{5v}{4a^2} \\ &- 3\frac{2r-a}{r^2}\log\frac{r}{a} + \frac{5}{2a}\log\frac{r}{l} + C_2 \end{aligned} \tag{57}$$

We see that the functions $m(v,r)$ and $\varphi(v,r)$ depend linearly of v, i.e. of time. There are two independent integration constants, C_1 and C_2. The expression for the ADM energy was found in [15]. The value of the energy is given by the value of the boundary term which has to be added to the canonical hamiltonian in order to have a well defined theory. It is given by

$$\Delta = -\delta H_b \ , \tag{58}$$

where

$$
\begin{aligned}
4G\Delta &= \frac{\sqrt{-g}}{g_{11}}(4Br'\delta r - 2\kappa\psi'\delta\psi + 12\kappa\psi'\delta\chi + 12\kappa\chi'\delta\psi) \\
&+ \frac{2}{\sqrt{-g}}\delta(\frac{-g}{g_{11}})(Arr' - \kappa\psi' + 6\kappa\chi') + \frac{2}{\sqrt{-g}}\Big(\frac{-g}{g_{11}}\Big)'(Ar\delta r - \kappa\delta\psi + 6\kappa\delta\chi) \\
&+ 4G\pi^{11}(2\delta g_{01} - \frac{g_{01}}{g_{11}}\delta g_{11}) + 4G\frac{g_{01}}{g_{11}}(\pi_r\delta r + \pi_\psi\delta\psi + \pi_\chi\delta\chi) \ . \qquad (59)
\end{aligned}
$$

δ denotes the variation in the chosen class of field configurations, described in more details in [15]. A and B are $A = 1 + \frac{6\kappa}{r^2}$, $B = 1 + \frac{6\kappa\psi}{r^2}$. Of course, in order to identify the real value of energy, we have to find it in a coordinate system which is asymptotically Minkowskian. As we have solved the equations for m and φ, we can now write the corrected values of the components of metric:

$$
\begin{aligned}
g_{00} &= -\Big(1 - \frac{a}{r} + \frac{\kappa m}{r} + 2\kappa(1 - \frac{a}{r})\varphi\Big) \\
g_{01} &= -\kappa\Big(\frac{m}{r-a} + \varphi\Big) \\
g_{11} &= \frac{r}{r-a} - \kappa\frac{mr}{(r-a)^2} \ , \qquad (60)
\end{aligned}
$$

so we see that, unlike the static case, the metric is not diagonal in the first order in κ.

In order to find a coordinate system $\tilde{t}$, $\tilde{r}$ in which the asymptotic values of the metric are

$$
\tilde{g}_{00} \to -1 + O(\frac{1}{L}) \ , \quad \tilde{g}_{01} \to 0 \qquad (61)
$$

(it is not really necessary to assume also $\tilde{g}_{11} \to 1$, as we are interested only in the value of the energy), we introduce the transformation of coordinates

$$
\tilde{t} = t + \kappa\alpha(t,r) \ , \quad \tilde{r} = r \ . \qquad (62)
$$

Under this transformation, the metric transforms as

$$
\begin{aligned}
\tilde{g}_{00} &= g_{00}(1 - 2\kappa\frac{\partial\alpha}{\partial t}) \\
\tilde{g}_{01} &= g_{01} - \kappa\frac{\partial\alpha}{\partial r}g_{00} \\
\tilde{g}_{11} &= g_{11} \ . \qquad (63)
\end{aligned}
$$

In accordance with the asymptotic relations ($^{asy]}$) the function α should be choosen in the form

$$
\alpha(t,r) = F_1 r + F_2 t + F_3 rt + F_4 t^2 \ , \qquad (64)
$$

where

$$
F_1 = \frac{5}{4a^2} + \frac{9}{aL} - \frac{5}{2a^2}\log\frac{L}{l} - \frac{5}{4aL}\log(\frac{L}{a} - 1) + \frac{LC_1}{L-a} + \frac{LC_2}{(L-a)^2} \qquad (65)
$$

$$F_2 = \frac{15}{8a^2} + \frac{45}{8aL} - \frac{5}{2a^2}\log\frac{L}{l} - \frac{5}{8aL}\log\frac{L}{a} + C_1 + \frac{C_2}{2(L-a)} \tag{66}$$

$$F_3 = -\frac{5}{4a^2L} \tag{67}$$

$$F_4 = \frac{5}{32a^2(L-a)} - \frac{5}{16a^2L} \ . \tag{68}$$

The coordinate transformation induces the following change in the boundary term:

$$4G\tilde{\Delta} = 4G\Delta - 2\kappa r\left(\frac{\partial\alpha}{\partial t}\delta(-\frac{g}{g_{11}}) + 2(-\frac{g}{g_{11}})\delta(\frac{\partial\alpha}{\partial t})\right) \ . \tag{69}$$

Introducing the obtained solutions for ψ, χ, g we get for the value of $\tilde{\Delta}$:

$$4G\tilde{\Delta} = -2\delta a + \kappa\Big(\frac{21}{4a^2} - \frac{11L}{2a^3} - \frac{C_2(a)}{L-a} - \frac{5}{a^2}\log\frac{L}{l} + \frac{5}{a^3}t\Big)\delta a \ . \tag{70}$$

The corresponding value of energy is

$$\tilde{H}_b = -\frac{1}{4G}\int 4G\tilde{\Delta} = M + \frac{\kappa}{4G}\Big(\frac{21}{4a} - \frac{11L}{4a^2} - \frac{5}{a}\log\frac{L}{l} + \frac{5}{2a^2}t + \int\frac{C_2}{L-a}da\Big) \tag{71}$$

The first term in ($^{adm]}$) is the classical mass of the black hole, while the second one is the quantum correction of the mass. We can take that $C_2 = 0$. One immediately notes the time-dependence of ADM mass, which is in agreement with radiation law of the black hole. Namely,

$$\frac{d\tilde{H}_b}{dt} = T_{uu}\mid_{r\to L} = -\frac{5}{192\pi a^2}. \tag{72}$$

The increase of the mass cooresponds to the fact that the outgoing flux is negative at large distances, e.g. that the black hole antievaporates. It is important to mention that the mass increases only if we consider large but finite volumes L. If we take the limit $L \to \infty$, the t-term in the expression for energy ($^{adm]}$) can be neglected in comparison with the larger terms proportional to $\log L$ and L, so we have the conservation of the energy of the whole system, $\dot{\tilde{H}}_b = 0$. Notice, that "mass function" $M(r,v) = M - \kappa\frac{m(r,v)}{2}$ satisfies the condition $\dot{M}(r,v) = -5/192\pi a^2$.

4 Apparent horizon and entropy

Apparent horizon is the boundary of the trapped surfaces. In 2D dilaton gravity it is defined by [18]

$$g^{\mu\nu}\partial_\mu r\partial_\nu r = 0 \ . \tag{73}$$

If we define the one-loop corrected null coordinates by

$$ds^2 = -e^{2\rho}d\bar{u}\,d\bar{v} \tag{74}$$

the condition ($^{trap]}$) is reduced to $\partial_{\bar u} r = 0$ and $\partial_{\bar v} r = 0$. We will take $\bar v = v = t + r_*$. The other null coordinate $\bar u$ can be found easily. The first step is to rewrite the metric ($^{eq:qs]}$) in the form

$$\begin{aligned} ds^2 &= -Fe^{2\kappa\varphi}(d\bar v - \frac{2}{F} e^{-\kappa\varphi} dr) d\bar v \\ &= -\frac{Fe^{2\kappa\varphi}}{\mu}(\mu d\bar v - \frac{2\mu}{F} e^{-\kappa\varphi} dr) d\bar v \ , \end{aligned} \tag{75}$$

where μ is the integration factor. Therefore, the conformal coordinate $\bar u$ satisfies

$$d\bar u = \mu d\bar v - \frac{2\mu}{F} e^{-\kappa\varphi} dr \ . \tag{76}$$

We will not solve the previous equation for $\bar u$, but just use it to find the position of the apparent horizon. From ($^{eq:dut]}$) we get

$$dr = \frac{1}{2} e^{\kappa\varphi} F (d\bar v - \frac{1}{\mu} d\bar u) \ . \tag{77}$$

The last equation, if we use $\partial_{\bar u} r = 0$ and $\partial_{\bar v} r = 0$ implies $e^{\kappa\varphi} F = 0$ on the horizon. This means that the equation of the apparent horizon is

$$(1 + \kappa\varphi)(1 - \frac{a}{r} + \frac{\kappa m}{r}) = 0 \ . \tag{78}$$

The position of the apparent horizon is found perturbatively taking $r_{AH} = a + \kappa r_1$, where r_1 is the first-order correction. From equation ($^{AH]}$) we get

$$r_{AH} = a - \kappa\Big(\frac{23}{4a} + \frac{5}{2a} \log\frac{a}{l} + \frac{1}{4a^2} \bar v\Big) \ . \tag{79}$$

The intersection point between the line of singularity and the apparent horizon is the endpoint of the Hawking radiation. It is given by

$$\bar u_{int} = \infty \ , \bar v_{int} = 4a^2\Big(\frac{a}{\kappa} - \frac{23}{4a} - \frac{5}{2a} \log\frac{a}{l}\Big) \approx \frac{4a^3}{\kappa} \ . \tag{80}$$

As we can take the $\bar v$-coordinate as the time, we see that the (anti)-evaporation of the black hole is very long but finite.

In order to calculate the entropy of the quantum corrected solution, we use the Wald technique [19]. Note that the conical singularity method is defined for static configurations only and therefore cannot be used here. In references [20, 21, 22] it was shown that for the lagrangians of the form $L = L(f_m, \nabla f_m, g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ (f_m are the matter fields) the entropy is given by

$$S = -2\pi \epsilon_{\alpha\beta}\, \epsilon_{\chi\delta} \frac{\partial L}{\partial R_{\alpha\beta\chi\delta}}\Big|_H \ ,$$

evaluated on the horizon. In our case we find

$$\begin{aligned} S &= \frac{\pi}{G}\Big(r^2 - \kappa(2\psi - 12\chi - 12\log r)\Big)\Big|_{AH} \\ &= \frac{\pi}{G}\Big(r^2 + \kappa(4\frac{r}{a} - 8\log\frac{r}{a} + 12\log\frac{r}{l} + \frac{1}{a}\bar{v})\Big)\Big|_{AH} \\ &= \frac{\pi}{G}\Big(a^2 - \kappa(\frac{15}{2} + 5\log\frac{a}{l} - \frac{\bar{v}}{2a})\Big) \ . \end{aligned} \tag{81}$$

Now, we will show that the entropy increases along the line of apparent horizon. This end we will find the equation for $\bar{u}$ coordinate. The integration factor, which we introduced in ([eq:dut]), is of the form

$$\mu = 1 + \kappa R(r) + \kappa V(\bar{v}) \ , \tag{82}$$

where $R(r)$ and $V(\bar{v})$ are unknown functions. If we introduce the ansatz ([ans]) in the condition of integrability of equation ([eq:dut]),

$$\frac{\partial \mu}{\partial r}\Big|_{\bar{v}} = -2\frac{\partial}{\partial \bar{v}}\Big(\frac{\mu}{F}e^{-\kappa\varphi}\Big)\Big|_r \ , \tag{83}$$

we obtain the following expressions

$$V(\bar{v}) = \alpha\bar{v} \ , \tag{84}$$

$$R(r) = -2\alpha r - \frac{1}{2a(r-a)} - \frac{4a^3\alpha + 5}{2a^2}\log(r-a) \ , \tag{85}$$

where α is the integration constant. On the other hand, if we start from

$$\frac{\partial \bar{u}}{\partial \bar{v}}\Big|_r = 1 + \kappa R(r) + \kappa V(\bar{v}) \ , \tag{86}$$

$$\frac{\partial \bar{u}}{\partial r}\Big|_{\bar{v}} = -\frac{2\mu}{F}e^{-\kappa\varphi} \tag{87}$$

we get

$$\bar{u} = \bar{v} + \kappa\bar{v}R(r) + \frac{1}{2}\kappa\alpha\bar{v}^2 + G(r) \ . \tag{88}$$

Therefore the function $G(r)$ is determined by equation

$$\begin{aligned} \frac{dG}{dr} &= -\frac{2r}{r-a}\Big[1 + \kappa\Big(-2\alpha r - \frac{1}{2a(r-a)} - \frac{5+4\alpha a^3}{2a^2}\log(r-a)\Big) \\ &- \kappa\Big(\frac{5}{ar} + \frac{3}{4r^2} + \frac{3}{r^2}\log\frac{r}{a} - \frac{5}{2a^2}\log\frac{r}{a} + C_1\Big) \\ &- \Big(\frac{\kappa}{r-a}(\frac{5r}{2a^2} + \frac{1}{2r} + \frac{11a}{4r^2} - 3\frac{2r-a}{r^2}\log\frac{r}{a} + \frac{5}{2a}\log\frac{r}{l}\Big)\Big] \ , \end{aligned} \tag{89}$$

which can easily be integrated. The derivative of the entropy along the apparent horizon is determined by

$$t^a \partial_a S = \Big(\frac{\partial}{\partial \bar{v}} + \frac{d\bar{u}_{AH}}{d\bar{v}} \frac{\partial}{\partial \bar{u}} \Big) S \ , \tag{90}$$

where t^a the is tangent vector of the apparent horizon. The expression $(^{eq:ah]})$ for the apparent horizon and $(^{V]}{}_{-}{}^{G]})$ give

$$t^a \partial_a S = \frac{\kappa \pi}{2aG} > 0 \ . \tag{91}$$

So, the entropy increases along the line of apparent horizon. This shows that the second law of thermodynamics is fulfilled in the framework of the SSG model.

5 Conclusions

In this paper we calculated the backreaction effects of the Hawking radiation in the Unruh state of the Schwarzchild black hole. The effect is discussed in the framework of the SSG model. The calculation was simplified using the formalism of auxiliary fields. It is shown that the definition of the Unruh state fixes the integration functions and that the corresonding EMT coincides with EMT calculated by other methods. The position of the apparent horizon is found and the evaporation of the black hole discussed. The obtained duration of the evaporation is large (proportional to $1/\kappa$). Unfortunately, at the intersecting point of the line of singularity and apparent horizon the singularity becomes naked, which prevents us from predicting the future evolution of the black hole. The discussion of the static remnant of the black hole is an interesting question and will be the subject of futher investigation. The entropy of the black hole-radiation system is obtained and it is shown it increases during the evolution. The quantum corrections of the energy of the system are calculated using the ADM procedure. We found that the flux of the radiation through the large spherical surface of the radius L is in accordance with the radiation law. In the limit $L \to \infty$ though, the energy of the whole system is conserved, as one would expect.

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